

#1

$$ay'' + by' + cy = g(x) (\neq 0)$$

→ { γραμμική 2ης τάξης
 σταθεροί συντελεστές
 μη-ομογενής.

γενική
 λύση

$$y(x) = y_0(x) + y_{\mu 0}(x)$$

$y_0(x)$ = γενική λύση της ομογενούς

$y_{\mu 0}(x)$ = ^(μια) μερική λύση της μη-ομογενούς

(α) $y'' - y = x^2 + 2x$

ομογενής $y'' - y = 0$

χαρακτηριστική $\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$
 εξίσωση

άρα: $y_0(x) = C_1 e^x + C_2 e^{-x} = C_1 e^x + C_2 e^{-x}$

για $\begin{cases} y_{\mu 0} = Ax^2 + Bx + \Gamma \\ y' = 2Ax + B \\ y'' = 2A \end{cases}$ γιατί $g(x) = x^2 + 2x$ (πολύωχη 2ος βαθμός)
 ε' αντικαθιστώντας στη δ.ε:

$$2A - Ax^2 - Bx - \Gamma = x^2 + 2x \Rightarrow$$

(x²) $-A = 1 \Rightarrow A = -1$

(x) $-B = 2 \Rightarrow B = -2$

(0) $2A - \Gamma = 0 \Rightarrow \Gamma = 2A = -2$

άρα $y_{\mu 0} = -x^2 - 2x - 2$

ε' $y = C_1 e^x + C_2 e^{-x} - x^2 - 2x - 2$

(β) $x \in \mathbb{R} \quad \tau^2 - 6\tau + 5 = 0 \Rightarrow \tau_{1,2} \begin{matrix} \nearrow 5 \\ \searrow 1 \end{matrix} \quad y_0 = C_1 e^{5x} + C_2 e^{4x}$

για $\begin{cases} y_{\mu 0} = Ae^{4x} \\ y' = 4Ae^{4x} \\ y'' = 16Ae^{4x} \end{cases}$ (γιατί $g(x) = e^{4x}$)

ε' δ.ε: $16Ae^{4x} - 24Ae^{4x} + 5Ae^{4x} = e^{4x}$
 $-3A = 1 \Rightarrow A = -\frac{1}{3}$

ε' $y_{\mu 0} = -\frac{1}{3} e^{4x}$

$$\textcircled{\gamma} \quad y_0 = \text{όπως (β)}, \quad \text{για} \quad y_{\mu 0} = A \sin 3x + B \cos 3x$$

$$y' = 3A \cos 3x - 3B \sin 3x$$

$$y'' = -9A \sin 3x - 9B \cos 3x$$

ε' στμ δ.ε:

$$-9A \sin 3x - 9B \cos 3x - 18A \cos 3x + 18B \sin 3x + 5A \sin 3x + 5B \cos 3x = 2 \cos 3x \Rightarrow$$

$$(-4A + 18B) \sin 3x + (-18A - 4B) \cos 3x = 2 \cos 3x$$

$$\begin{cases} -4A + 18B = 0 \Rightarrow B = \frac{4}{18} A \\ -18A - 4B = 2 \end{cases}$$

$$\Rightarrow -18A + \frac{4}{18} A = 2 \Rightarrow$$

$$\frac{-324A + 4A}{18} = 2 \Rightarrow \frac{-320A}{18} = 2$$

$$B = \frac{4}{18} \left(-\frac{9}{80} \right) = -\frac{9}{360} = -\frac{1}{40} \Rightarrow A = -\frac{36}{320} = -\frac{9}{80}$$

$$\text{άρα} \quad y_{\mu 0} = -\frac{9}{80} \sin 3x - \frac{1}{40} \cos 3x$$

$$\textcircled{\delta} \quad y = y_0 + y_{\mu 0}^{(1)} + y_{\mu 0}^{(2)} \quad \text{για} \quad y_{\mu 0}^{(1)} : y'' - 6y' + 5y = e^{4x} \quad (\text{δς (β)})$$

$$y_0 = \text{όπως} \quad \text{πριν} \quad y_{\mu 0}^{(2)} : y'' - 6y' + 5y = 2 \cos 3x \quad (\text{δς (γ)})$$

$$\textcircled{\epsilon} \quad y_0 = \text{όπως} \quad \text{πριν}$$

(παράγωγο με x)

$$y_{\mu 0} = A e^{5x}$$

$$\text{γιατί το } 5 \text{ παρά τις } x \text{ ε:}$$

$$t^2 - 6t + 5 = 0$$

$$\begin{cases} y' = A e^{5x} + 5A x e^{5x} \\ y'' = 5A e^{5x} + 5A e^{5x} + 25A x e^{5x} = 10A e^{5x} + 25A x e^{5x} \end{cases}$$

ε' στμ δ.ε:

$$10A e^{5x} + 25A x e^{5x} - 6A e^{5x} - 30A x e^{5x} + 5A e^{5x} = e^{5x}$$

$$4A e^{5x} = e^{5x} \Rightarrow 4A = 1 \Rightarrow A = \frac{1}{4}$$

$$\text{Άρα} \quad y_{\mu 0} = \frac{1}{4} e^{5x}$$

$$(f) \quad y'' - 6y' + 5y = \cos^2 x = \frac{1 + \cos 2x}{2} = \frac{1}{2} + \frac{1}{2} \cos 2x$$

$$y = y_0 + y_{\text{hom}}^{(1)} + y_{\text{hom}}^{(2)} : \quad y_{\text{hom}}^{(1)} \text{ για } : y'' - 6y' + 5y = \frac{1}{2}$$

$(y_0 = 0 \text{ πως } \pi \text{ πιν})$
 $y_{\text{hom}}^{(1)} = A \quad (\text{0 πως } (\alpha))$

$$y_{\text{hom}}^{(2)} \text{ για } : y'' - 6y' + 5y = \frac{1}{2} \cos 2x$$

$$y_{\text{hom}}^{(2)} = \frac{A \cos 2x + B \sin 2x}{(\text{0 πως } (\gamma))}$$

$$(n) \quad y'' + 25 = \sin 3x + 7 \cos 3x$$

$$y_0 : x \in \tau^2 + 25 = 0 \Rightarrow \tau = \pm 5i = 0 \pm 5i$$

αρχ

$$y_0 = C_1 e^{5x} \cos 5x + C_2 e^{5x} \sin 5x = C_1 \cos 5x + C_2 \sin 5x$$

$$y_{\text{hom}} = A \sin 3x + B \cos 3x$$

(αναγωγή, 0 πως (γ))

$$\text{γιατί } g(x) = \sin 3x + 7 \cos 3x$$

$$(δ) \quad x \in \tau^4 - 6\tau^2 + 5 = 0$$

$$\text{αν } w = \tau^2 \text{ τότε}$$

$$w^2 - 6w + 5 = 0 \Rightarrow \begin{cases} w_{1,2} = 5, 1 \\ w_1 = \tau^2 = 5 \Rightarrow \tau = \pm \sqrt{5} \\ w_2 = \tau^2 = 1 \Rightarrow \tau = \pm 1 \end{cases}$$

αρχ

$$y_{\text{hom}} = C_1 e^{\sqrt{5}x} + C_2 e^{-\sqrt{5}x} + C_3 e^x + C_4 e^{-x}$$

$$\text{Για } y_{\text{hom}} = Ax^2 + Bx + \Gamma$$

(αναγωγή 0 πως (α))

$$\text{γιατί } g(x) = x^2 + x + 3$$

$$(i) \quad y'' - 2y' + y = x e^{3x}$$

$$y_{\text{hom}} : x \in \tau^2 - 2\tau + 1 = 0 \Rightarrow (\tau - 1)^2 = 0$$

$\tau = 1 \text{ διπλή}$

$$y_{\text{hom}} = C_1 e^{1x} + C_2 x e^{1x}$$

↑
διπλή

$$y_{\text{part}} = (Ax+B)e^{3x}$$

γιατί

$$g(x) = xe^{3x} \quad \text{Ποζώνυμο 1ου βαθμού}$$

$$y' = (Axe^{3x} + Be^{3x})' = Ae^{3x} + 3Axe^{3x} + 3Be^{3x}$$

$$y'' = \underbrace{3Ae^{3x} + 3Ae^{3x}}_{6Ae^{3x}} + 9Axe^{3x} + 9Be^{3x}$$

ε'τσι δε:

$$\underline{6Ae^{3x}} + 9Axe^{3x} + \underline{9Be^{3x}} - \underline{2Ae^{3x}} - \underline{6Axe^{3x}} - \underline{6Be^{3x}} + \underline{Axe^{3x}} + \underline{Be^{3x}} = xe^{3x}$$

$$\begin{cases} 4A + 4B = 0 \Rightarrow A = -B \quad \eta \quad B = -A \\ 4A = 1 \Rightarrow A = \frac{1}{4} \quad \eta \quad B = -\frac{1}{4} \end{cases}$$

$$\begin{cases} 4A = 1 \Rightarrow A = \frac{1}{4} \quad \eta \quad B = -\frac{1}{4} \end{cases}$$

$$\alpha\pi\alpha: y_{\text{part}} = \left(\frac{1}{4}x - \frac{1}{4}\right)e^{3x}$$

$$(K) \quad y''' - 2y'' - 4y' + 8y = \frac{e^{3x} + e^{-3x}}{2}$$

$$y_0: x \in \quad \tau^3 - 2\tau^2 - 4\tau + 8 = 0 \Rightarrow \tau^2(\tau - 2) - 4(\tau - 2) = 0$$

$$\Rightarrow (\tau^2 - 4)(\tau - 2) = 0 \Rightarrow (\tau - 2)(\tau + 2)(\tau - 2) = 0$$

$$\Rightarrow \tau = -2$$

$$\tau = \pm \sqrt{4} \eta$$

$$\alpha\pi\alpha \quad y_0 = c_1 e^{-2x} + c_2 e^{2x} + c_3 e^{2x} x$$

$$y_{\text{part}} = y_{\text{part}}^{(1)} + y_{\text{part}}^{(2)}$$

$$\begin{cases} y_{\text{part}}^{(1)}: y''' - 2y'' - 4y' + 8y = \frac{1}{2}e^{3x} \\ y_{\text{part}}^{(1)} = Ae^{3x} \end{cases}$$

$$\begin{cases} y_{\text{part}}^{(2)}: y''' - 2y'' - 4y' + 8y = \frac{1}{2}e^{-3x} \\ y_{\text{part}}^{(2)} = Ae^{-3x} \end{cases}$$

αναλόγως όπως (α)

$$\mathcal{L}(x^n) = \frac{n!}{s^{n+1}}$$

(#2) (α) $\mathcal{L}(f) = \mathcal{L}(x^2) + 5\mathcal{L}(x) + \mathcal{L}(7) = \frac{2!}{s^3} + 5\frac{1}{s^2} + 7\frac{1}{s} = \frac{2}{s^3} + \frac{5}{s^2} + \frac{7}{s}$

(β) $\mathcal{L}(\cos 6x) = \frac{s}{s^2 + 6^2}$

οπότε $\mathcal{L}(e^{5x} \cos 6x) = \frac{(s-5)}{(s-5)^2 + 6^2}$

γιατί εάν $\mathcal{L}(f) = F(s)$

$\mathcal{L}(e^{\alpha x} f) = F(s-\alpha)$

(όπου s αντικαθίσταται με $s-\alpha$)

(γ) $e^{-x}(e^{2x} + 4e^{-x})^2 = e^{-x}(e^{2x})^2 + 8e^{2x}e^{-x} + 16(e^{-x})^2$
 $= e^{-x}(e^{4x} + 8e^x + 16e^{-2x})$
 $= e^{3x} + 8 + 16e^{-3x}$

οπότε:

$\mathcal{L}(f_3) = \mathcal{L}(e^{3x}) + \mathcal{L}(8) + 16\mathcal{L}(e^{-3x})$
 $= \frac{1}{s-3} + 8/s + 16\frac{1}{s+3}$

γιατί:
 $\mathcal{L}(e^{\alpha x}) = \frac{1}{s-\alpha}$

(δ) $\sin(3x + \frac{\pi}{6}) = \sin 3x \cdot \cos \frac{\pi}{6} + \sin \frac{\pi}{6} \cos 3x$
 $= \frac{\sqrt{3}}{2} \sin 3x + \frac{1}{2} \cos 3x$

(γιατί $\sin(\alpha+\beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha$)

άρα $\mathcal{L}(\sin(3x + \frac{\pi}{6})) = \frac{\sqrt{3}}{2} \mathcal{L}(\sin 3x) + \frac{1}{2} \mathcal{L}(\cos 3x)$
 $= \frac{\sqrt{3}}{2} \frac{3}{s^2 + 3^2} + \frac{1}{2} \frac{s}{s^2 + 3^2}$

(ε) χρήση του (δ)

$\mathcal{L}(e^{5x} \sin(3x + \frac{\pi}{6})) = \frac{\sqrt{3}}{2} \frac{3}{(s-5)^2 + 3^2} + \frac{1}{2} \frac{s-5}{(s-5)^2 + 3^2}$

(στ) $\mathcal{L}(x \cos \alpha x) = ; \quad \mathcal{L}(\cos \alpha x) = \frac{s}{s^2 + \alpha^2}$

π' $\mathcal{L}(x \cos \alpha x) = - \left(\frac{s}{s^2 + \alpha^2} \right)' = - \frac{s'(s^2 + \alpha^2) - s(s^2 + \alpha^2)'}{(s^2 + \alpha^2)^2}$
 $= \frac{-(s^2 + \alpha^2 - 2s^2)}{(s^2 + \alpha^2)^2} = \frac{s^2 - \alpha^2}{(s^2 + \alpha^2)^2}$

γιατί: εάν $\mathcal{L}(f) = F(s) \Rightarrow \mathcal{L}(xf) = (-1) \frac{dF(s)}{ds}$

(#3) (α) $L(y) = \frac{s}{s^2+25} \Rightarrow y = \cos 5x$

(β) $L(y) = \frac{s+3}{(s+3)^2+25} \Rightarrow y = e^{-3x} \cos 5x$

όπου $s \rightarrow$
το $(s-(-3))$

(γ) $L(y) = \frac{4-s}{s^2+16} = \frac{1}{s^2+16} - \frac{s}{s^2+16} = \frac{1}{4} \cdot \frac{4}{s^2+16} - \frac{s}{s^2+16}$
 $\Rightarrow y = \frac{1}{4} \sin 4x - \cos 4x$

(δ) $L(y) = \frac{2s+1}{s^2+s} = \frac{2s+1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \quad \left. \vphantom{\frac{2s+1}{s(s+1)}} \right\} \Rightarrow$
 (αμφάκη κλάσματα)

$A(s+1) + Bs = 2s+1$
 $s=0 \quad A=1$
 $s=-1 \quad -B=-1 \Rightarrow B=1$

$\text{αμφ. } L(y) = \frac{1}{s} + \frac{1}{s+1}$

$\Rightarrow y = 1 + e^{-x}$

(ε) $L(y) = \frac{1}{s^3-s} = \frac{1}{s(s^2-1)} = \frac{1}{s(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{\Gamma}{s+1}$

$A(s-1)(s+1) + B(s+1)s + \Gamma s(s-1) = 1$

για $s=1 \quad 2B=1 \Rightarrow B=1/2$

$s=-1 \quad 2\Gamma=1 \Rightarrow \Gamma=1/2$

$s=0 \quad -A=1 \Rightarrow A=-1$

οπότε $L(y) = -\frac{1}{s} + \frac{1}{s-1} + \frac{1}{s+1} \Rightarrow$

$y = -1 + e^x + e^{-x}$

#4

$$\text{16x01a: } \int h(y') = sh(y) - y(0) \quad (7)$$

$$\mathcal{L}(h(y'')) = s^2 h(y) - sy(0) - y'(0)$$

$$(a) \mathcal{L}(y'' + 9y) = \mathcal{L}(8) \Rightarrow \mathcal{L}(y') + 9\mathcal{L}(y) = \frac{8}{s}$$

$$(y(0)=1) \Rightarrow s\mathcal{L}(y) - y(0)' + 9\mathcal{L}(y) = \frac{8}{s} \Rightarrow$$

$$(s+9)\mathcal{L}(y) = \frac{8}{s} + 1 = \frac{8+s}{s} \Rightarrow$$

$$\mathcal{L}(y) = \frac{8+s}{s(s+9)} \xrightarrow{\text{partial fractions}} \frac{A}{s} + \frac{B}{s+9} \Rightarrow$$

$$A(s+9) + Bs = 8+s$$

$$\left. \begin{array}{l} \text{για } s=0 \\ s=-9 \end{array} \right\} \begin{array}{l} 9A=8 \Rightarrow A=\frac{8}{9} \\ -9B=-1 \Rightarrow B=\frac{1}{9} \end{array} \Rightarrow \mathcal{L}(y) = \frac{8}{9} \cdot \frac{1}{s} + \frac{1}{9} \cdot \frac{1}{s+9}$$

$$\text{αρα } y = \frac{8}{9} + \frac{1}{9} e^{-9x}$$

$$(b) \mathcal{L}(y'' + 8y) = \mathcal{L}(e^{2x}) \Rightarrow \mathcal{L}(y') + 8\mathcal{L}(y) = \frac{1}{s-2}$$

$$\Rightarrow s\mathcal{L}(y) - y(0)' + 8\mathcal{L}(y) = \frac{1}{s-2} \Rightarrow (s+8)\mathcal{L}(y) = \frac{1}{s-2}$$

$$\Rightarrow \mathcal{L}(y) = \frac{1}{(s-2)(s+8)} = \frac{A}{s-2} + \frac{B}{s+8} \quad A, B = ?$$

$$\text{οπότε } y = Ae^{2x} + Be^{-8x}$$

$$(c) \mathcal{L}(y'' - 7y' + 12y) = 0 \Rightarrow$$

$$s^2 \mathcal{L}(y) - sy(0)' - y'(0)' - 7s\mathcal{L}(y) + 7y(0)' + 12\mathcal{L}(y) = 0$$

$$(s^2 - 7s + 12)\mathcal{L}(y) = s - 7 \Rightarrow \mathcal{L}(y) = \frac{s-7}{s^2 - 7s + 12} = \frac{s-7}{(s-3)(s-4)}$$

$$= \frac{A}{s-3} + \frac{B}{s-4} \quad (A, B = ?)$$

$$\text{αρα } y = Ae^{3x} + Be^{4x}$$

$$(d) \mathcal{L}(y'' + 25y) = \mathcal{L}(e^{3x}) \Rightarrow s^2 \mathcal{L}(y) - \cancel{sy(0)} - \cancel{y'(0)} + 25\mathcal{L}(y) = \frac{1}{s-3}$$

$$\Rightarrow (s^2 + 25)\mathcal{L}(y) = \frac{1}{s-3} \Rightarrow$$

$$\mathcal{L}(y) = \frac{1}{(s-3)(s^2+25)} = \frac{A}{s-3} + \frac{Bs+\Gamma}{s^2+25} \Rightarrow$$

$$A(s^2+25) + (Bs+\Gamma)(s-3) = 1$$

$$s=3 \quad 34A = 1 \Rightarrow A = \frac{1}{34}$$

$$s=0 \quad 25A - 3\Gamma = 1 \Rightarrow -3\Gamma = 1 - \frac{25}{34} = \frac{9}{34} \Rightarrow \Gamma = -\frac{3}{34}$$

$$s=1 \quad 26A - 2B - 2\Gamma = 1 \Rightarrow \frac{26}{34} - 2B + \frac{6}{34} = 1 \Rightarrow B = -\frac{1}{34}$$

$$\text{Άρα } \mathcal{L}(y) = \frac{1}{34} \frac{1}{s-3} + \frac{-\frac{1}{34}s - \frac{3}{34}}{s^2+25} \quad (\text{το σπάμε σε 2 κλάσματα})$$

$$= \frac{1}{34} \frac{1}{s-3} - \frac{1}{34} \frac{s}{s^2+25} - \frac{3}{34} \cdot \frac{1}{s} \frac{s}{s^2+25}$$

$$\Rightarrow y = \frac{1}{34} e^{3x} - \frac{1}{34} \cos 5x - \frac{3}{34} \cdot \frac{1}{5} \sin 5x$$

$$(e) \mathcal{L}(y'') + 3\mathcal{L}(y') + 2\mathcal{L}(y) = 5\mathcal{L}(x) + \mathcal{L}(3) \Rightarrow$$

$$s^2 \mathcal{L}(y) - \cancel{sy(0)} - \cancel{y'(0)} + 3s\mathcal{L}(y) - 3\cancel{y(0)} + 2\mathcal{L}(y) = \frac{5}{s^2} + \frac{3}{s} =$$

$$(s^2 + 3s + 2)\mathcal{L}(y) = \frac{5+3s}{s^2} \Rightarrow \mathcal{L}(y) = \frac{5+3s}{s^2(s^2+3s+2)} =$$

$$= \frac{s+3s}{s^2(s+1)(s+2)} \quad \text{✓}$$

$$L(y) = \frac{A}{s} + \frac{B}{s^2} + \frac{\Gamma}{s+1} + \frac{\Delta}{s+2}$$

$A, B, \Gamma, \Delta =$
(κατά τα γνωστά)

$$\Rightarrow y = A + Bx + \Gamma e^{-x} + \Delta e^{-2x}$$

(#5) (α)

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$$

πίνακας συντελεστών συστήματος.

για τις ιδιοτιμές: $\begin{vmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(2-\lambda) - 2 = 0 \Rightarrow \dots$

$$\Rightarrow \lambda^2 - 5\lambda + 4 = 0 \Rightarrow \lambda \begin{matrix} \nearrow 4 \\ \searrow 1 \end{matrix}$$

Ιδιοδιανύσματα:

$\lambda = 4$
 $u_1 = \begin{pmatrix} x \\ y \end{pmatrix}$ $A \begin{pmatrix} x \\ y \end{pmatrix} = 4 \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 4 \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{matrix} 3x + 2y = 4x \\ x + 2y = 4y \end{matrix}$

$\Rightarrow$ (αόριστο) $\begin{matrix} 2y = x \\ (x = 2y) \end{matrix}$ κ' εάν $y = 1 \Rightarrow x = 2$
δηλ. $u_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda = 1$
 $u_2 = \begin{pmatrix} x \\ y \end{pmatrix}$ $A \begin{pmatrix} x \\ y \end{pmatrix} = 1 \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{matrix} 3x + 2y = x \\ x + 2y = y \end{matrix}$

$\Rightarrow x = -y$
κ' εάν $y = 1 \Rightarrow x = -1$

δηλ. $u_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Γενική λύση $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{4x} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{1 \cdot x} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ η'

$$\begin{cases} x_1 = 2c_1 e^{4x} - c_2 e^x \\ x_2 = c_1 e^{4x} + c_2 e^x \end{cases}$$